constructed clocks with their difference in gravitational potential $\Delta\psi = gh$, the clock at the lower potential having the longer observed period.

Furthermore, since time measurements could be made with the help of the period of light corresponding to any given spectral line, we can evidently regard different atoms of the same substance as furnishing the identically constructed clocks necessary for the validity of the above relation. Hence, making use of (79.3) together with the relation between the period and wave-length of light, we are at once led to the conclusion that there should be an observed shift $\delta\lambda$ of the approximate amount

$$\delta\lambda = \lambda\frac{\Delta\psi}{c^2} \tag{79.4}$$

in the wave-length λ of light which passes through a difference in gravitational potential of amount $\Delta\psi$, in travelling from the point of origin to that where the observation is made. The observational verification of this result will be more particularly mentioned in § 83 (c), in connexion with the three so-called crucial tests of relativity.

(c) The clock paradox. The foregoing relation between the rate of a clock and its gravitational potential has also been found to furnish the solution for a well-known paradox, which can arise when the behaviour of clocks is treated in accordance with the principles of special relativity without making due allowance for the principles of the general theory.

Consider two identically constructed clocks A and B, originally together and at rest, and let a force F_1 be then applied for a short time to clock B giving it the velocity u with which it then travels away from A at a constant rate for a time which is long compared with that necessary for the acceleration. At the end of this time let a second force F_2 be applied in the reverse direction which brings B to rest and starts it back towards A with the reversed velocity $-u$. And finally, when it has returned to the neighbourhood of A, let the clock B be brought once more to rest by the action of a third force F_3.

Since by hypothesis the time intervals necessary for the acceleration and deceleration of clock B are made negligibly short compared with the time of travel at the constant velocity u, we can then write, in accordance with the decreased rate of a moving clock given by the special theory of relativity (see § 9),

$$\Delta t_A = \frac{\Delta t_B}{\sqrt{(1-u^2/c^2)}} = \Delta t_B\left(1+\frac{1}{2}\frac{u^2}{c^2}+...\right) \tag{79.5}$$

as an expression connecting the measurements Δt_A and Δt_B—on the two different clocks—of the elapsed time necessary for the clock B to move out from A and return. In accordance with this expression we are thus led to the definite conclusion that clock B would register a smaller number of divisions than clock A at the end of the indicated experiment.

At first sight, nevertheless, this conclusion—obtained quite correctly from the special theory of relativity—appears incompatible with the idea of the relativity of all motion, since it should then be equally as acceptable to regard B as the clock which remains at rest and consider A as moving away with the velocity $-u$ and returning with the velocity $+u$. And taking A as the moving clock, it then seems as if A should be the clock that registers the smaller number of divisions.

The apparent paradox is, however, readily solved with the help of the general theory of relativity, if we do not neglect the actual lack of symmetry between the treatment given to the clock A which was at no time subjected to any force, and that given to the clock B which was subjected to the successive forces F_1, F_2, and F_3 when the relative motion of the clocks was changed. To preserve this same state of affairs in a valid description of the experiment, taking A as the moving clock and B as the one which remains at rest, we may assume that the changes in the relative motion of the two clocks are produced by the temporary introduction of homogeneous gravitational fields, which are allowed to act freely on A in such a way as to produce the desired changes in velocity without A experiencing any force, and in such a way as to necessitate the action of the same forces on B as before in order to maintain it at rest. This then gives us a valid description of the *identical* experiment in the new language, and we can easily calculate the relation which would now be expected between the two time measurements Δt_A and Δt_B.

To do this, let us first put

$$\Delta t_A = \tau_A+\tau'_A+\tau''_A+\tau'''_A \tag{79.6}$$

and

$$\Delta t_B = \tau_B+\tau'_B+\tau''_B+\tau'''_B, \tag{79.7}$$

where τ_A and τ_B are the time measurements referred to the two clocks during which the clock A is now regarded as having the uniform velocity u, and τ'_A, τ''_A, τ'''_A and τ'_B, τ''_B, τ'''_B are the times needed for the three changes in the velocity of A which are brought about at the beginning, middle, and end of the experiment by the temporary introduction of an appropriate gravitational field as mentioned above. And let us take these latter intervals as very short compared with the time during which A is in uniform motion, in correspondence with the previous description of the experiment.

Since the clock A is now the one which moves, we can in the first place write in accordance with the special theory of relativity to the desired order of precision

$$\tau_A = \tau_B\left(1-\frac{1}{2}\frac{u^2}{c^2}+\ldots\right), \tag{79.8}$$

in contrast to the previous relation (79.5) where B was taken as the moving clock. Furthermore, since the two clocks will be at practically the same potential when the gravitational fields are introduced at the beginning and end of the experiment, we can evidently write with sufficient precision

$$\tau'_A = \tau'_B \quad \text{and} \quad \tau'''_A = \tau'''_B. \tag{79.9}$$

On the other hand, when the gravitational field is introduced at the middle of the experiment to produce the necessary reversal in the motion of A, the two clocks will be at a great distance from each other, and we must evidently write in accordance with our previous treatment

$$\tau''_A = \tau''_B\left(1+\frac{\Delta\psi}{c^2}\right), \tag{79.10}$$

where $\Delta\psi$ is their difference in gravitational potential at that time.

This difference in potential, however, is given in terms of the distance between the two clocks h and the gravitational acceleration g by the simple expression

$$\Delta\psi = hg.$$

Furthermore, we can evidently put

$$h = \tfrac{1}{2}u\tau_B,$$

since $2h$ is the total distance travelled at the speed u, and can write

$$g = \frac{2u}{\tau''_B}$$

since $2u$ is the total change in velocity in the time τ''_B. Substituting these three expressions we can then write (79.10) in the more useful form

$$\tau''_A = \tau''_B + \tau_B \frac{u^2}{c^2}, \tag{79.11}$$

and combining this equation with the previous equations (79.6–9) we obtain

$$\begin{aligned}\Delta t_A &= \tau_B\left(1 - \frac{1}{2}\frac{u^2}{c^2} + \ldots\right) + \tau'_B + \tau''_B + \tau_B\frac{u^2}{c^2} + \tau'''_B \\ &= \tau_B\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots\right) + \tau'_B + \tau''_B + \tau'''_B,\end{aligned}$$

or to our order of approximation, since the primed quantities are very short compared with τ_B,

$$\Delta t_A = \Delta t_B\left(1 + \frac{1}{2}\frac{u^2}{c^2}\right). \tag{79.12}$$

Comparing this result with the earlier equation (79.5), we now see that whether we consider A or B to be the clock which moves we obtain the same expression for the relative readings of the two clocks, to the order of approximation that has been employed. The treatment of the problem without approximation would involve the full apparatus of the general theory of relativity.

The solution thus provided for the well-known clock paradox of the special theory gives a specially illuminating example of the justification for regarding all kinds of motion as relative, that has been made possible by the adoption of the general theory of relativity.

A similar treatment can also be given with entire success to the difference in rate between a clock placed at the centre of a rotating platform and a second clock fixed to the periphery of the platform. If the platform is taken as rotating, the peripheral clock will be regarded as having a slower rate than the central clock because of its velocity of motion. On the other hand, if the platform is taken as at rest and the remainder of the universe as rotating in the opposite direction, the slower rate of the peripheral clock will be ascribed to its position of lower gravitational potential corresponding to the gravitational interpretation which would then be given to centrifugal action. The general idea of the relativity of all kinds of motion will thus again be preserved, since we can with equal success treat the platform or the remainder of the universe as subject to the rotation.