Eq. (8.34), with $x_0 = 0$ for simplicity, we now have

$$\frac{1}{(2/m)^{1/2}} \int_0^x \frac{dx}{(E + Fx)^{1/2}} = t$$

or

$$\frac{2}{F}(E + Fx)^{1/2} - \frac{2}{F} E^{1/2} = \left(\frac{2}{m}\right)^{1/2} t.$$

Solving for x, we get

$$x = \frac{1}{2}\left(\frac{F}{m}\right) t^2 + \left(\frac{2E}{m}\right)^{1/2} t.$$

But $F/m = a$, and since $E = \frac{1}{2}mv^2 + Fx$ is the total energy, we have that at $t = 0$, when $x = 0$, the energy E is all kinetic and is equal to $\frac{1}{2}mv_0^2$. Thus $2E/m = v_0^2$, and we finally obtain for x, $x = \frac{1}{2}at^2 + v_0 t$, which is the same expression we obtained before, in Eq. (5.11), with $x_0 = 0$ and $t_0 = 0$. This problem is sufficiently simple for it to be more easily solved by the methods of Chapter 5. We have presented it here mainly as an illustration of the techniques for solving the equation of motion using the principle of energy.

8.10 Motion under Conservative Central Forces

In the case of a central force, when E_p depends only on the distance r, Eq. (8.28) becomes

$$E = \tfrac{1}{2}mv^2 + E_p(r), \tag{8.35}$$

from which it is possible to determine the velocity at any distance. In many cases the function $E_p(r)$ decreases in absolute value when r increases. Then, at very large distances from the center, $E_p(r)$ is negligible and the magnitude of the velocity is constant and is independent of the direction of motion. This is the principle we applied in Example 7.16 when, in Fig. 7–28, we indicated that the final velocity of the receding particle at B was the same as its initial velocity at A.

Note that, when we are dealing with motion under the influence of central forces, there are two conservation theorems. One is the conservation of angular momentum, discussed in Section 7.13, and the other is the conservation of energy, expressed by Eq. (8.35). When we use polar coordinates r and θ, and remember that the components of the velocity are $v_r = dr/dt$ and $v_\theta = r\,d\theta/dt$, we may write, according to Eq. (5.63),

$$v^2 = v_r^2 + v_\theta^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2.$$

But from the principle of conservation of angular momentum, using Eq. (7.35), $L = mr^2\,d\theta/dt$, we have that

$$r^2\left(\frac{d\theta}{dt}\right)^2 = \frac{L^2}{(mr)^2},$$

where L is the constant angular momentum. Therefore

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{(mr)^2}.$$

Introducing this result into Eq. (8.35), we have

$$E = \tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} + E_p(r). \tag{8.36}$$

This expression closely resembles Eq. (8.32) for rectilinear motion, with velocity dr/dt, if we assume that, insofar as the radial motion is concerned, the particle moves under an "effective" potential energy

$$E_{p,\text{eff}}(r) = \frac{L^2}{2mr^2} + E_p(r). \tag{8.37}$$

The first term is called the *centrifugal* potential, $E_{p,c}(r) = L^2/2mr^2$, because the "force" associated with it, using Eq. (8.25), is $F_c = -\partial E_{p,c}/\partial r = L^2/mr^3$ and, being positive, is pointing away from the origin; that is, it is centrifugal. Of course no centrifugal force is acting on the particle, except the one that may be due to the real potential $E_p(r)$, in the event that it is repulsive and the centrifugal "force" F_c is just a useful mathematical concept. Physically this concept describes the tendency of the particle, according to the law of inertia, to move in a straight line and thus avoid moving in a curve. Introducing Eq. (8.37) into Eq. (8.36), we have

$$E = \tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + E_{p,\text{eff}}(r),$$

and solving for dr/dt, we obtain

$$\frac{dr}{dt} = \left\{\frac{2}{m}\left[E - E_{p,\text{eff}}(r)\right]\right\}^{1/2}, \tag{8.38}$$

which is formally identical to Eq. (8.33) for rectilinear motion. Separating the variables r and t and integrating (setting $t_0 = 0$ for convenience), we obtain

$$\int_{r_0}^{r} \frac{dr}{\{(2/m)[E - E_{p,\text{eff}}(r)]\}^{1/2}} = \int_0^t dt = t, \tag{8.39}$$

which gives us the distance r as a function of time [that is, $r(t)$], and therefore we have the solution of our dynamical problem corresponding to radial motion.

When we solve the expression for the angular momentum, $L = mr^2\, d\theta/dt$ for $d\theta/dt$, we have

$$\frac{d\theta}{dt} = \frac{L}{mr^2}. \tag{8.40}$$

Then when we introduce $r(t)$ as obtained from Eq. (8.39) into Eq. (8.40), we express L/mr^2 as a function of time, and when we integrate we have

$$\int_{\theta_0}^{\theta} d\theta = \int_0^t \frac{L}{mr^2}\,dt \qquad \text{or} \qquad \theta = \theta_0 + \int_0^t \frac{L}{mr^2}\,dt. \tag{8.41}$$

This gives θ as a function of time; that is, $\theta(t)$. In this way we can solve the problem completely, giving both the radial and the angular motions as functions of time.

Sometimes, however, we are more interested in the equation of the path. Combining Eqs. (8.38) and (8.40) through division, we may write

$$\frac{dr}{d\theta} = \frac{\{(2/m)[E - E_{p,\mathrm{eff}}(r)]\}^{1/2}}{L/mr^2} \tag{8.42}$$

or, separating the variables r and θ and integrating,

$$\int_{r_0}^{r} \frac{dr}{(m/L)r^2\{(2/m)[E - E_{p,\mathrm{eff}}(r)]\}^{1/2}} = \int_{\theta_0}^{\theta} d\theta = \theta - \theta_0. \tag{8.43}$$

This expression relating r to θ gives the equation of the path in polar coordinates. Conversely, if we know the equation of the path, so that we can compute $dr/d\theta$, Eq. (8.42) allows us to compute the potential energy and then the force.

This section has illustrated how the principles of conservation of angular momentum and of energy allow us to solve for the motion of a particle acted on by a central force. By now the student will have recognized the fact that these principles are not mathematical curiosities, but real and effective tools for solving dynamical problems. We must note that when the motion is due to a central force, the conservation of energy is not enough to solve the problem. It is also necessary to use the conservation of angular momentum. In the case of rectilinear motion, the conservation of energy *is* sufficient to solve the problem. This is because energy is a scalar quantity, and may not be used to determine the direction of motion, while in rectilinear motion, the direction is fixed from the outset.

Finally, let us make it especially clear that the principles of conservation of angular momentum and of energy, as used in this chapter, are properties associated with an individual particle under the special circumstances of its motion, and there is no direct relation to the possible conservation of total energy of the universe. This subject will be discussed in more detail in the next chapter.

8.11 Discussion of Potential Energy Curves

The graphs representing $E_p(x)$ versus x in rectilinear or one-dimensional problems and $E_p(r)$ versus r in central force problems are very useful in helping one to understand the motion of a particle, even without solving the equation of motion. In Fig. 8–18 we have illustrated a possible potential energy curve for one-dimensional motion. When we use the first of Eqs. (8.23), the force on the particle for any