PROBLEMS

PROBLEM 1. Find a complete integral of the Hamilton–Jacobi equation for motion of a particle in a field $U = \alpha/r - Fz$ (a combination of a Coulomb field and a uniform field), and find a conserved function of the co-ordinates and momenta that is specific to this motion.

SOLUTION. The field is of the type (48.15), with $a(\xi) = \alpha - \frac{1}{4}F\xi^2$, $b(\eta) = \alpha + \frac{1}{4}F\eta^2$. The complete integral of the Hamilton–Jacobi equation is given by (48.16) with these functions $a(\xi)$ and $b(\eta)$. To determine the significance of the constant β, we write the equations

$$2\xi p_\xi^2 + ma(\xi) - mE\xi + \tfrac{1}{4}p_\phi^2/\xi = \beta,$$

$$2\eta p_\eta^2 + mb(\eta) - mE\eta + \tfrac{1}{4}p_\phi^2/\eta = -\beta.$$

Subtracting, and expressing the momenta $p_\xi = \partial S/\partial\xi$ and $p_\eta = \partial S/\partial\eta$ in terms of the momenta $p_\rho = \partial S/\partial\rho$ and $p_z = \partial S/\partial z$ in cylindrical co-ordinates, we obtain after a simple calculation

$$\beta = -m\left[\frac{\alpha z}{r} + \frac{p_\rho}{m}(zp_\rho - \rho p_z) + \frac{p_\phi^2}{m\rho^2}z\right] - \tfrac{1}{2}mF\rho^2.$$

The expression in the brackets is an integral of the motion that is specific to the pure Coulomb field (the z-component of the vector (15.17)).

PROBLEM 2. The same as Problem 1, but for a field $U = \alpha_1/r_1 + \alpha_2/r_2$ (the Coulomb field of two fixed points at a distance 2σ apart).

SOLUTION. This field is of the type (48.21), with $a(\xi) = (\alpha_1 + \alpha_2)\xi/\sigma$, $b(\eta) = (\alpha_1 - \alpha_2)\eta/\sigma$. The action $S(\xi, \eta, \varphi, t)$ is obtained by substituting these expressions in (48.22). The significance of the constant β is found in a manner similar to that in Problem 1; in this case it expresses the conservation of the quantity

$$\beta = \sigma^2\left(p^2 + \frac{p_\phi^2}{\rho^2}\right) - M^2 + 2m\sigma(\alpha_1\cos\theta_1 + \alpha_2\cos\theta_2),$$

$$M^2 = (\mathbf{r}\times\mathbf{p})^2$$

$$= p^2z^2 + p_z^2\rho^2 + \frac{r^2p_\phi^2}{\rho^2} - 2z\rho p_z p_\rho,$$

and θ_1 and θ_2 are the angles shown in Fig. 55.

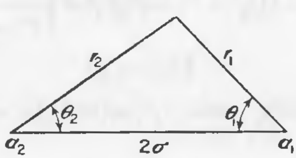

FIG. 55

§49. Adiabatic invariants

Let us consider a mechanical system executing a finite motion in one dimension and characterised by some parameter λ which specifies the properties of the system or of the external field in which it is placed,† and let us suppose that λ varies slowly (*adiabatically*) with time as the result of some external action; by a "slow" variation we mean one in which λ varies only slightly during the period T of the motion:

$$T\,d\lambda/dt \ll \lambda. \tag{49.1}$$

† To simplify the formulae, we assume that there is only one such parameter, but all the results remain valid for any number of parameters.

If λ were constant, the system would be closed and would execute a strictly periodic motion with a constant energy E and a fixed period $T(E)$. When the parameter λ is variable, the system is not closed and its energy is not conserved. However, since λ is assumed to vary only slowly, the rate of change $\dot{E}$ of the energy will also be small. If this rate is averaged over the period T and the "rapid" oscillations of its value are thereby smoothed out, the resulting value $\dot{E}$ determines the rate of steady slow variation of the energy of the system, and this rate will be proportional to the rate of change $\dot{\lambda}$ of the parameter. In other words, the slowly varying quantity E, taken in this sense, will behave as some function of λ. The dependence of E on λ can be expressed as the constancy of some combination of E and λ. This quantity, which remains constant during the motion of a system with slowly varying parameters, is called an *adiabatic invariant*.

Let $H(q, p; \lambda)$ be the Hamiltonian of the system, which depends on the parameter λ. According to formula (40.5), the rate of change of the energy of the system is

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}. \tag{49.2}$$

The expression on the right depends not only on the slowly varying quantity λ but also on the rapidly varying quantities q and p. To ascertain the steady variation of the energy we must, according to the above discussion, average (49.2) over the period of the motion. Since λ and therefore λ vary only slowly, we can take $\dot{\lambda}$ outside the averaging:

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{\mathrm{d}\lambda}{\mathrm{d}t}\overline{\frac{\partial H}{\partial \lambda}}, \tag{49.3}$$

and in the function $\partial H/\partial\lambda$ being averaged we can regard only q and p, and not λ, as variable. In other words, the averaging is taken over the motion which would occur if λ remained constant.

The averaging may be explicitly written

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^T \frac{\partial H}{\partial \lambda}\,\mathrm{d}t.$$

According to Hamilton's equation $\dot{q} = \partial H/\partial p$, or $\mathrm{d}t = \mathrm{d}q \div (\partial H/\partial p)$. The integration with respect to time can therefore be replaced by one with respect to the co-ordinate, with the period T written as

$$T = \int_0^T \mathrm{d}t = \oint \mathrm{d}q \div (\partial H/\partial p); \tag{49.4}$$

here the $\oint$ sign denotes an integration over the complete range of variation ("there and back") of the co-ordinate during the period.† Thus (49.3) becomes

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{\mathrm{d}\lambda}{\mathrm{d}t}\frac{\oint (\partial H/\partial\lambda)\,\mathrm{d}q/(\partial H/\partial p)}{\oint \mathrm{d}q/(\partial H/\partial p)}. \tag{49.5}$$

As has already been mentioned, the integrations in this formula must be taken over the path for a given constant value of λ. Along such a path the Hamiltonian has a constant value E, and the momentum is a definite function of the variable co-ordinate q and of the two independent constant parameters E and λ. Putting therefore $p = p(q; E, \lambda)$ and differentiating with respect to λ the equation $H(q,p,\lambda) = E$, we have $\partial H/\partial\lambda + (\partial H/\partial p)(\partial p/\partial\lambda) = 0$, or

$$\frac{\partial H/\partial\lambda}{\partial H/\partial p} = -\frac{\partial p}{\partial\lambda}.$$

Substituting this in the numerator of (49.5) and writing the integrand in the denominator as $\partial p/\partial E$, we obtain

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = -\frac{\mathrm{d}\lambda}{\mathrm{d}t}\frac{\oint (\partial p/\partial\lambda)\,\mathrm{d}q}{\oint (\partial p/\partial E)\,\mathrm{d}q}$$

or

$$\oint \left(\frac{\partial p}{\partial E}\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} + \frac{\partial p}{\partial\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)\mathrm{d}q = 0.$$

Finally, this may be written as

$$\overline{\mathrm{d}I/\mathrm{d}t} = 0, \tag{49.6}$$

where

$$I \equiv \oint p\,\mathrm{d}q/2\pi, \tag{49.7}$$

the integral being taken over the path for given E and λ. This shows that, in the approximation here considered, I remains constant when the parameter λ varies, i.e. I is an adiabatic invariant.

The quantity I is a function of the energy of the system (and of the parameter λ). The partial derivative with respect to energy determines the period of the motion: from (49.4),

$$2\pi\frac{\partial I}{\partial E} = \oint \frac{\partial p}{\partial E}\,\mathrm{d}q = T \tag{49.8}$$

† If the motion of the system is a rotation, and the co-ordinate q is an angle of rotation ϕ, the integration with respect to ϕ must be taken over a "complete rotation", i.e. from 0 to 2π.

or

$$\partial E/\partial I = \omega, \tag{49.9}$$

where $\omega = 2\pi/T$ is the vibration frequency of the system.

The integral (49.7) has a geometrical significance in terms of the phase path of the system. In the case considered (one degree of freedom), the phase space reduces to a two-dimensional space (i.e. a plane) with co-ordinates p, q, and the phase path of a system executing a periodic motion is a closed curve in the plane. The integral (49.7) taken round this curve is the area enclosed. It can be written also as the area integral

$$I = \iint \mathrm{d}p\, \mathrm{d}q/2\pi. \tag{49.10}$$

As an example, let us determine the adiabatic invariant for a one-dimensional oscillator. The Hamiltonian is

$$H = \tfrac{1}{2}p^2/m + \tfrac{1}{2}m\omega^2 q^2, \tag{49.11}$$

where ω is the eigenfrequency of the oscillator. The equation of the phase path is given by the law of conservation of energy $H(p, q) = E$. The path is an ellipse with semi-axes $\sqrt{(2mE)}$ and $\sqrt{(2E/m\omega^2)}$, and its area, divided by 2π, is

$$I = E/\omega. \tag{49.12}$$

The adiabatic invariance of I signifies that, when the parameters of the oscillator vary slowly, the energy is proportional to the frequency.

§50. Canonical variables

Now let the parameter λ be constant, so that the system in question is closed. Let us effect a canonical transformation of the variables q and p, taking I as the new "momentum". The generating function is the abbreviated action S_0, expressed as a function of q and I. For S_0 is defined as the integral

$$S_0(q, E; \lambda) = \int p(q, E; \lambda)\, \mathrm{d}q, \tag{50.1}$$

taken for a given energy E and parameter λ. For a closed system, however, I is a function of the energy alone, and so S_0 can equally well be written as a function $S_0(q, I; \lambda)$, and the partial derivative $(\partial S_0/\partial q)_E$ is the same as the derivative $(\partial S_0/\partial q)_I$ for constant I. Hence

$$p = \partial S_0(q, I; \lambda)/\partial q, \tag{50.2}$$

corresponding to the first of the formulae (45.8) for a canonical transformation. The second of these formulae gives the new "co-ordinate", which we denote by w:

$$w = \partial S_0(q, I; \lambda)/\partial I. \tag{50.3}$$

The variables I and w are called *canonical variables*; I is called the *action variable* and w the *angle variable*.